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Normal-Ordered Products with Grassmann Operators in Supersymmetry

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Normal-ordered products with corresponding contractions for Grassmann operators that appear in supersymmetry are defined. It is shown, using the case of chiral superfields, that these normal-ordered products are useful in practical manipulations. As a demonstration, they are used to simplify evaluation of superfield propagators, functional differentiations, and integrations by parts.

1. INTRODUCTION

In supersymmetry (Wess and Zumino, 1974a, b; Volkov and Akulov, 1972) superfields which are functions of superspace coordinates $z=(x, \theta, \bar{\theta})$ always have finite series expansions in Grassmann variables θ and $\bar{\theta}$ (Salam and Strathdee, 1974a, b; Wess and Bagger, 1983). [In notations, metric, and various definitions, we closely follow Wess and Bagger (1983).] Hence, for example, when differential operators act on some superfield quantity, it would be desirable to have differentiations with respect to θ and $\bar{\theta}$ done separately from differentiations with respect to x, which in turn would render the whole thing properly expressed as a power series in θ and $\bar{\theta}$. This task can be accomplished naturally with the help of normal-ordered products containing "elementary" Grassmann operators, where the elementary Grassmann operators are defined as θ_α and $\bar{\theta}_{\dot{\alpha}}$, and supersymmetry differential operators containing $\partial/\partial\theta_\alpha$ and $\partial/\partial\bar{\theta}_{\dot{\alpha}}$ ($\alpha, \dot{\alpha}=1, 2$):

$$\begin{aligned} Q_\alpha &= \partial/\partial\theta^\alpha + \sigma^m_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}} P_m \\ \bar{Q}_{\dot{\alpha}} &= -\partial/\partial\bar{\theta}^{\dot{\alpha}} - \theta^\alpha \sigma^m_{\alpha\dot{\alpha}} P_m \\ D_\alpha &= \partial/\partial\theta^\alpha - \sigma^m_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}} P_m \\ \bar{D}_{\dot{\alpha}} &= -\partial/\partial\bar{\theta}^{\dot{\alpha}} + \theta^\alpha \sigma^m_{\alpha\dot{\alpha}} P_m \end{aligned} \tag{1}$$

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Here $P_m = -i\,\partial/\partial x^m$. While Q and $\bar{Q}$ are differential operators from ($N = 1$) superalgebra and are used to deduce the transformation laws for superfields, D and $\bar{D}$ are used to formulate superfield equations of motion and to impose the convariant constraints on superfields.

According to our definition of elementary Grassmann operators, they are either covariant [as in (1)] or contravariant vectors. The relationships between them are as follows:

$$A^\alpha = \varepsilon^{\alpha\beta} A_\beta, \qquad \bar{A}^{\dot\alpha} = \varepsilon^{\dot\alpha\dot\beta} \bar{A}^{+}_{\dot\beta} \tag{2}$$

Here A_α ($\bar{A}_{\dot\alpha}$) denotes θ_α, Q_α, and D_α ($\bar\theta_{\dot\alpha}$, $\bar{Q}_{\dot\alpha}$, and $\bar{D}_{\dot\alpha}$), while $\varepsilon^{ab} = -\varepsilon_{ab}$ ($a = \alpha, \dot\alpha$; $b = \beta, \dot\beta$) is the antisymmetric Levi-Civita symbol, with $\varepsilon_{ab}\varepsilon^{bc} = \delta^c_a$. Scalar products $AB = A^\alpha B_\alpha = -A_\alpha B^\alpha$ and $\bar{A}\bar{B} = \bar{A}_{\dot\alpha}\bar{B}^{\dot\alpha} = -\bar{A}^{\dot\alpha}\bar{B}_{\dot\alpha}$ satisfy

$$AB = BA + \varepsilon^{\alpha\beta}\{A_\beta, B_\alpha\}; \qquad \bar{A}\bar{B} = \bar{B}\bar{A} + \varepsilon_{\dot\alpha\dot\beta}\{\bar{A}^{\dot\beta}, \bar{B}^{\dot\alpha}\} \tag{3}$$

Next we list the values for various anticommutators among Grassmann operators that will be needed later. (For simplicity, from now on, an elementary Grassmann operator is referred to simply as a Grassmann operator.) They are:

$$\begin{aligned} &\{Q_\alpha, \theta_\beta\} = \{D_\alpha, \theta_\beta\} = \varepsilon_{\beta\alpha} \\ &\{\bar{Q}_{\dot\alpha}, \bar\theta_{\dot\beta}\} = \{\bar{D}_{\dot\alpha}, \bar\theta_{\dot\beta}\} = \varepsilon_{\dot\alpha\dot\beta} \end{aligned} \tag{4}$$

$$\{\bar{Q}_\alpha, \bar{Q}_{\dot\alpha}\} = -\{D_\alpha, \bar{D}_{\dot\alpha}\} = -2\sigma^m_{\alpha\dot\alpha} P_m \tag{5}$$

$$\begin{aligned} &\{\theta_\alpha, \theta_\beta\} = \{\theta_\alpha, \bar\theta_{\dot\beta}\} = \{\bar\theta_{\dot\alpha}, \bar\theta_{\dot\beta}\} = 0 \\ &\{Q_\alpha, \bar\theta_{\dot\beta}\} = \{D_\alpha, \bar\theta_{\dot\beta}\} = \{\bar{Q}_{\dot\alpha}, \theta_\beta\} = \{\bar{D}_{\dot\alpha}, \theta_\beta\} = 0 \\ &\{D_\alpha, Q_\beta\} = \{D_\alpha, \bar{Q}_{\dot\beta}\} = \{\bar{D}_{\dot\alpha}, Q_\beta\} = \{\bar{D}_{\dot\alpha}, \bar{Q}_{\dot\beta}\} = 0 \end{aligned} \tag{6a}$$

$$\begin{aligned} \{Q_\alpha, \theta'_\beta\} &= \{D_\alpha, \theta'_\beta\} = \{Q'_\alpha, \theta_\beta\} \\ &= \{D'_\alpha, \theta_\beta\} = \cdots = \{\bar{D}_{\dot\alpha}, \bar{Q}'_{\dot\beta}\} = \{\bar{D}'_{\dot\alpha}, \bar{Q}_{\dot\beta}\} = 0 \end{aligned} \tag{6b}$$

where in (6b), for example, $D'_\alpha = \partial/\partial\theta'_\alpha - \sigma^m_{\alpha\dot\alpha}\bar\theta^{\dot\alpha} P'_m$, $P'_m = -i\,\partial/\partial x'^m$. [Unprimed and primed operators are to be associated with $z = (x, \theta, \bar\theta)$ and $z' = (x', \theta', \bar\theta')$, respectively.] Clearly, any other anticommutator, such as $\{Q_\alpha, \theta^\beta\}$, can be easily obtained with the help of (2). Anticommutators (4)-(6) will be the backbone of the contractions between various Grassmann operators that are necessary to reduce products of ordinary or normal-ordered products into the superposition of normal-ordered products of Grassmann operators.

In Section 2 we give the definitions of normal-ordered products and contractions of Grassmann operators. Some simple applications of these normal-ordered products for chiral superfields are given in Section 3. Section 4 is devoted to a discussion of the results and the conclusion.

2. NORMAL-ORDERED PRODUCTS AND CONTRACTIONS OF GRASSMANN OPERATORS

The definition of the normal-ordered product of Grassmann operators is very simple. Let us take n Grassmann operators (θ, $\bar{\theta}$, Q, $\bar{Q}$, D, $\bar{D}$, θ', $\bar{\theta}', \ldots, \theta'', \ldots$), and for simplicity denote them as $G(1)$, $G(2), \ldots, G(n)$. Then the symbol $:G(1)G(2)\cdots G(n):$ defines their normal product; i.e., the product in which by definition Grassmann operators $G(1), G(2), \ldots, G(n)$ behave as if they were ordinary anticommuting Grassmann variables.

As far as one can tell, the definition of the normal-ordered product here is very similar to the definition of the normal-ordered product in quantum field theory involving fermion creation and annihilation operators. In fact, the usual rules for normal-ordered products from quantum field theory apply here also. Specifically, the distribution law and the permutation rule (factors inside the normal-ordered product may be permutted as if all the anticommutators were equal to zero) are valid here also. However, the difference comes because, unlike for quantum fields, here the index of Grassmann operator G_a can assume only two values: $a = 1, 2$ ($a = \alpha, \dot{\alpha}$). The permutation rule then allows Grassmann operator G to appear no more than twice in the normal-ordered product, since $G_a G_b G_c = 0$ (as a consequence of $\{G_a, G_b\} = 0$).

Next we address the question of contractions between various Grassmann operators. Now the difference with contractions between quantum Fermi fields will be encountered. Namely, while in quantum field theory the anticommutators between annihilation (or creation) fermion operators are zero, here, as seen from (5), the anticommutators between the differential Grassmann operators are not always zero; this will cause the existence of contractions between these differential Grassmann operators, complicating the situation somewhat.

In general, two Grassmann operators $G(1)$ and $G(2)$ can have two contractions $\underline{G(1)G}(2)$ and $\underline{G(2)G}(1)$ defined as

$$G(1)G(2) = :G(1)G(2): + \underline{G(1)G}(2) \tag{7a}$$

$$G(2)G(1) = :G(2)G(1): + \underline{G(2)G}(1) \tag{7b}$$

Since by permutation rule $:\{G(1), G(2)\}: = 0$, we have that $\underline{G(1)G}(2)$ and $\underline{G(2)G}(1)$ must satisfy the constraint

$$\underline{G(1)G}(2) + \underline{G(2)G}(1) = \{G(1), G(2)\} \tag{7c}$$

For our Grassmann operators all the contractions are easily calculable, and, consistent with (4)–(6) and (7c), they fall into the following three categories:

$$\text{I.} \qquad \underline{G(1)G}(2) = 0, \qquad \underline{G(2)G}(1) = \{G(1), G(2)\} \tag{8}$$

$$\text{II.} \qquad \underbracket{G(1)G}(2) = \underbracket{G(2)G}(1) = \tfrac{1}{2}\{G(1), G(2)\} \tag{9}$$

$$\text{III.} \qquad \underbracket{G(1)G}(2) = \underbracket{G(2)G}(1) = \{G(1), G(2)\} = 0 \tag{10}$$

As we see, under the contraction sign $\sqcup$, the Grassmann operators may be interchanged (with a positive sign) in categories II and III of contractions, while in category I they cannot be interchanged at all. Next we list the specific contractions. From category I they are

$$\begin{aligned} &\underbracket{\theta_\alpha Q}_\beta = \underbracket{\theta_\alpha D}_\beta = 0, \qquad \underbracket{Q_\alpha \theta}_\beta = \underbracket{D_\alpha \theta}_\beta = \varepsilon_{\beta\alpha} \\ &\underbracket{\bar{\theta}_{\dot{\alpha}} \bar{Q}}_{\dot{\beta}} = \underbracket{\bar{\theta}_{\dot{\alpha}} \bar{D}}_{\dot{\beta}} = 0, \qquad \underbracket{\bar{Q}_{\dot{\alpha}} \bar{\theta}}_{\dot{\beta}} = \underbracket{\bar{D}_{\dot{\alpha}} \bar{\theta}}_{\dot{\beta}} = \varepsilon_{\dot{\alpha}\dot{\beta}} \end{aligned} \tag{11}$$

The contractions of category II are between few differential operators that carry undotted and dotted indices, respectively:

$$\underbracket{D_\alpha \bar{D}}_{\dot{\alpha}} = \underbracket{\bar{D}_{\dot{\alpha}} D}_\alpha = -\underbracket{Q_\alpha \bar{Q}}_{\dot{\alpha}} = -\underbracket{\bar{Q}_{\dot{\alpha}} Q}_\alpha = \sigma^m_{\alpha\dot{\alpha}} P_m \tag{12}$$

Contractions (12) are of category II because, for example, we have

$$\{\partial/\partial\bar{\theta}^{\dot{\alpha}}, D_\alpha\} = \{\partial/\partial\theta^\alpha, \bar{D}_{\dot{\alpha}}\} = \sigma^m_{\alpha\dot{\alpha}} P_m \tag{13}$$

The contractions of category III, which are numerically zero regardless of the order of contracting Grassmann operators, are the most numerous, since they appear whenever the anticommutator between two Grassmann operators vanishes [compare with (10)]. Consequently, all contractions between Grassmann operators appearing in vanishing anticommutators (6a) and (6b) are zero and are of category III:

$$\underbracket{\theta_\alpha \theta}_\beta = \cdots = \underbracket{\bar{D}_{\dot{\alpha}} \bar{Q}}_{\dot{\beta}} = \underbracket{Q_\alpha \theta'}_\beta = \cdots = \underbracket{\bar{D}'_{\dot{\alpha}} \bar{Q}}_{\dot{\beta}} = 0 \tag{14}$$

Let us also mention that contractions between contravariant and covariant vector-Grassmann operators are obtained with the help of (2), as in the example

$$\underbracket{D^\alpha \bar{D}}_{\dot{\alpha}} = \varepsilon^{\alpha\beta} \underbracket{D_\beta \bar{D}}_{\dot{\alpha}} = \varepsilon^{\alpha\beta} \sigma^m_{\beta\dot{\alpha}} P_m$$

The practical significance of all of these contractions is that they can be used in the normal-ordered product expansions. For example, an ordinary product of Grassmann operators can be written as a superposition of normal-ordered products according to a familiar rule:

$$[\cdots G(i) \cdots] = :\left[\cdots \left(G(i) + \sum_j \underbracket{G(i)G}(j) \frac{\partial}{\partial G(j)}\right) \cdots\right]: \tag{15}$$

where the summation over j also means summation over indices α and $\dot{\alpha}$. Similar rules can be devised for reducing a product of normal-ordered products into a superposition of normal-ordered products, which is illustrated in the simple example

$$:[\cdots G(i)\cdots]::[\cdots G(j)\cdots]:$$
$$=:\left[\cdots\left(G(i)+\sum_k{}' \underbrace{G(i)G(k)}\frac{\partial}{\partial G(k)}\right)\cdots\right][\cdots G(j)\cdots]: \quad (16)$$

where the prime on the summation sign indicates that $\partial/\partial G(k)$ acts on the Grassmann operators in the second bracket only.

3. SOME SIMPLE APPLICATIONS

As in our applications of the normal-ordered products of Grassmann operators, we shall deal exclusively with chiral superfields; here, following Salam and Strathdee (1974a, b), Wess and Bagger (1983), and Mohapatra (1986), we briefly review their properties. The chiral superfield Φ is required to satisfy the supersymmetric invaraint constraint:

$$\bar{D}_{\dot{\alpha}}\Phi=0 \quad (17a)$$

Since $\bar{D}_{\dot{\alpha}}y^m=0$ and $\bar{D}_{\dot{\alpha}}\theta=0$, where $y^m=x^m+i\theta\sigma^m\bar{\theta}$, we see that Φ can be expressed alternatively in (y, θ) and $(x, \theta, \bar{\theta})$ representations with the following components:

$$\begin{aligned}\Phi&=A(y)+\sqrt{2}\theta\psi(y)+\theta^2F(y)\\ &\equiv\exp(-\theta\sigma^m\bar{\theta}P_m)[A(x)+\sqrt{2}\theta\psi(x)+\theta^2F(x)]\\ &=A(x)-\theta\sigma^m\bar{\theta}P_mA(x)+\tfrac{1}{4}\theta^2\bar{\theta}^2\Box A(x)\\ &\quad+\sqrt{2}\theta\psi(x)+(1/\sqrt{2})\theta^2P_m\psi(x)\sigma^m\bar{\theta}+\theta^2F(x)\end{aligned} \quad (17b)$$

Here A is a scalar, ψ a spinor, and F an auxiliary field of dimension 2. The antichiral superfield is simply Φ^+, and, because $D_\alpha y^{+m}=0$ and $D_\alpha\bar{\theta}=0$ ($y^{+m}=x^m-i\theta\sigma^m\bar{\theta}$), it satisfies

$$D_\alpha\Phi^+=0 \quad (18a)$$

Its power series expansion is obtained from (17b) by conjugation:

$$\begin{aligned}\Phi^+&=A^*(y^+)+\sqrt{2}\bar{\theta}\bar{\psi}(y^+)+\bar{\theta}^2F^*(y^+)\\ &\equiv\exp(\theta\sigma^m\bar{\theta}P_m)[A^*(x)+\sqrt{2}\bar{\theta}\bar{\psi}(x)+\bar{\theta}^2F^*(x)]\\ &=A^*(x)+\theta\sigma^m\bar{\theta}P_mA^*(x)+\tfrac{1}{4}\theta^2\bar{\theta}^2\Box A^*(x)\\ &\quad+\sqrt{2}\bar{\theta}\bar{\psi}(x)-(1/\sqrt{2})\bar{\theta}^2\theta\sigma^m\bar{\theta}P_m\bar{\psi}(x)+\bar{\theta}^2F^*(x)\end{aligned} \quad (18b)$$

Because of the absence of differential Grassmann operators, both expressions are trival cases of expansions in terms of the normal-ordered Grassmann operators θ and $\bar{\theta}$.

Let us now expand in terms of the normal-ordered Grassmann operators expressions containing θ and $\bar{\theta}$ and differential operators D and $\bar{D}$. We start with $D^2\,\delta(\theta-\theta')$ and $\bar{D}^2\,\delta(\bar{\theta}-\bar{\theta}')$, where $\delta(\theta)=\theta^2$ and $\delta(\bar{\theta})=\bar{\theta}^2$, and, for later reference, $\delta(z)=\delta(x)\,\delta(\theta)\,\delta(\bar{\theta})$. With straightforward applications of equation (15) and appropriate contractions, one obtains

$$-\tfrac{1}{4}D^2\delta(\theta-\theta')=:\exp[-D(\theta-\theta')]: \tag{19a}$$

$$-\tfrac{1}{4}\bar{D}^2\delta(\bar{\theta}-\bar{\theta}')=:\exp[-\bar{D}(\bar{\theta}-\bar{\theta}')]: \tag{19b}$$

We immediately demonstrate the usefulness of relations (19) by noticing that, consistent with (17a), we have the following functional derivative for the chiral fields (Wess and Bagger, 1983):

$$\delta\Phi(z')/\delta\Phi(z)=\delta(\theta-\theta')\delta(y-y') \tag{20a}$$

Equation (20a) does not contain $\delta(z-z')$ explicitly. However, expanding $\delta(y-y')$ and utilizing (19b), we obtain

$$\begin{aligned}\delta(\theta-\theta')\,\delta(y-y')&=\delta(\theta-\theta')\exp[-\theta\sigma^m(\theta-\bar{\theta}')P_m]\delta(x-x')\\&=\delta(\theta-\theta')\,:\exp[-\bar{D}(\bar{\theta}-\bar{\theta}')]:\,\delta(x-x')\\&=-\tfrac{1}{4}\bar{D}^2\delta(z-z')\end{aligned} \tag{20b}$$

which now contains $\delta(z-z')$ explicitly. As we see, the notion of normal-ordered Grassmann operators allows us to derive relation (20b) directly rather than to find it from the variation of Φ under full superspace integrations (Wess and Bagger, 1983).

Next, because of (17), we have, for example, that $:\exp[-\bar{D}(\bar{\theta}-\bar{\theta}')]:\Phi(z)=\Phi(z)$. Consequently, we have also

$$-\tfrac{1}{4}D^2\delta(\theta-\theta')\Phi^+(z)=\Phi^+(z) \tag{21a}$$

$$-\tfrac{1}{4}\bar{D}^2\delta(\bar{\theta}-\bar{\theta}')\Phi(z)=\Phi(z) \tag{21b}$$

Relations (21) are very useful when carrying out integrations by parts. Because the projection operator on chiral fields is $(\bar{D}^2D^2/16\Box)$ (Wess and Bagger, 1983), we use (21) in the two examples

$$\begin{aligned}\int\delta(\bar{\theta})\Phi^2(z)\,d^8z&=\int\delta(\bar{\theta})\Phi(z)\frac{\bar{D}^2D^2}{16\Box}\Phi(z)\,d^8z\\&=-\frac{1}{4}\int\Phi(z)\frac{D^2}{\Box}\Phi(z)\end{aligned} \tag{22}$$

$$\int\delta(\bar{\theta})\Phi(z)\,\bar{D}^2\Phi(z)\,d^8z=-4\int\Phi^2(z)\,d^8z \tag{23}$$

where $d^8z=d^4x\,d^2\theta\,d^2\bar{\theta}\equiv d^4x\,d^4\theta$. Case (23) is particularly noteworthy, since, in view of (17a), one would naively expect (23) to vanish.

Let us now give an example where only the contractions between differential Grassmann operators occur. The example used is to rewrite the projection operator for chiral fields in a normal-ordered form as follows:

$$\Phi = \frac{1}{16}\frac{\bar{D}^2 D^2}{\Box}\Phi = \frac{1}{16}\frac{[\bar{D}^2, D^2]}{\Box}\Phi$$
$$= \left(1 - \frac{D\sigma^m \bar{D} P_m}{2\Box}\right)\Phi = \frac{:D\sigma^m \bar{D}:\, P_m}{2P^2} \tag{24}$$

Now $\Phi(x, \theta, \bar{\theta})$, according to (17b), when expanded in terms of θ and $\bar{\theta}$ is the generator of supermultiplet (independent) components (Salam and Strathdee, 1974a, b). Therefore, the normal-ordered operator in (24) cannot mix the different components, so we must have

$$\left(\frac{1}{2P^2}:D\sigma^m\bar{D}:\,P_m - 1\right)\left[A(x) - \theta\sigma^m\bar{\theta}P_m A(x) - \frac{1}{4}\theta^2\bar{\theta}^2 P^2 A(x)\right] = 0$$
$$\left(\frac{1}{2P^2}:D\sigma^m\bar{D}:\,P_m - 1\right)\left[\sqrt{2}\theta\psi(x) + \frac{1}{\sqrt{2}}\theta^2 P_m \psi\sigma^m\bar{\theta}\right] = 0 \tag{25}$$
$$\left(\frac{1}{2P^2}:D\sigma^m\bar{D}:\,P_m - 1\right)\theta^2 F(x) = 0$$

One verifies relations (25) with normal-ordered product expansions according to (16).

We list some other illustrative cases of results of normal-ordered products of Grassmann operators:

$$:\exp(\alpha\bar{D}\bar{\theta})::\exp(-\bar{D}\bar{\theta}): = :\exp(-\bar{D}\bar{\theta}): \tag{26a}$$

$$:\exp(\bar{D}\bar{\theta})::\exp(\bar{D}\bar{\theta}): = :\exp(3\bar{D}\bar{\theta}): \tag{26b}$$

$$:\exp(-D\theta):\,\delta(\theta) = 0 \tag{26c}$$

$$\delta(\theta)\,:\exp(-D\theta): = \delta(\theta) \tag{26d}$$

where α is an arbitrary constant.

When evaluating various chiral free-field propagators, in addition to relations (19), one might also find the following relations very useful:

$$\bar{D}^2 D^2 \delta(\theta-\theta')\delta(\bar{\theta}-\bar{\theta}')$$
$$= 16\,:\exp\{-[D(\theta-\theta') + \bar{D}(\bar{\theta}-\bar{\theta}') + (\theta-\theta')\sigma^m(\bar{\theta}-\bar{\theta}')P_m]\}: \tag{27a}$$

$$D^2\bar{D}^2 \delta(\theta-\theta')\delta(\bar{\theta}-\bar{\theta}')$$
$$= 16\,:\exp\{-[D(\theta-\theta') + \bar{D}(\bar{\theta}-\bar{\theta}') - (\theta-\theta')\sigma^m(\bar{\theta}-\bar{\theta}')P_m]\}: \tag{27b}$$

whose evaluations involve contractions between D and θ and between $\bar{D}$ and $\bar{\theta}$ (the first two terms in the square brackets), as well as between D and $\bar{D}$ (the third term in the square brackets), respectively. In propagator evaluations, relations (19) and (27) act on $\delta(x-x')$; now one should take into account that within the normal-ordered product, differential operators D_α znd $\bar{D}_{\dot{\alpha}}$ can be replaced with $-_\alpha(\sigma^m\bar{\theta})P_m$ and $(\theta\sigma^m)_{\dot{\alpha}}P_m$, respectively. So, for example, from (19a) we obtain

$$D^2\delta(\theta-\theta')\delta(x-x')=-4\exp[-i(\theta-\theta')\sigma^m\bar{\theta}\partial_m]\delta(x-x')$$

which is now the same as in Wess and Bagger (1983).

In general and when deriving Feynman rules for supergraphs, integrations by parts are very important. In this connection one would like to know how, in general, a product of Grassmann operators associated with $z=(x,\theta,\bar{\theta})$ can be transformed into a product associated with $z'=(x',\theta',\bar{\theta}')$, when acting on $\delta(z-z')$. In simpler cases one is not surprised to find that

$$D_\alpha\delta(z-z')=-D'_\alpha\delta(z-z') \tag{28a}$$

$$\bar{D}_{\dot{\alpha}}\delta(z-z')=-\bar{D}'_{\dot{\alpha}}\delta(z-z') \tag{28b}$$

$$D^2\delta(z-z')=D'^2\delta(z-z') \tag{28c}$$

$$\bar{D}^2\delta(z-z')=\bar{D}'^2\delta(z-z') \tag{28d}$$

These relations can be verified with normal-order product expansions, as in the example

$$\begin{aligned}D^2\delta(z-z')&=-4:\exp[(\theta-\theta')\sigma^m\bar{\theta}P_m]:\ \delta(\bar{\theta}-\bar{\theta}')\delta(x-x')\\&=-4:\exp[-D'(\theta'-\theta)]:\ \delta(\bar{\theta}-\bar{\theta}')\delta(x-x')\\&=D'^2\delta(z-z')\end{aligned}$$

However, using normal-order expressions (27), one derives explicitly that

$$D^2\bar{D}^2\delta(z-z')=\bar{D}'^2D'^2\delta(z-z') \tag{29a}$$

$$\bar{D}^2D^2\delta(z-z')=D'^2\bar{D}'^2\delta(z-z') \tag{29b}$$

i.e., $D^2\bar{D}^2\delta(z-z')\neq D'^2\bar{D}'^2\delta(z-z')$, etc. Relations (28c), (28d), and (29) can also be derived by utilizing the fact that contractions between unprimed and primed Grassmann operators (associated with z and z', respectively) are always zero [see relation (14)]. Then, for example, relation (29a) can be derived simply as

$$\begin{aligned}(29\text{a})&=-D^2\bar{D}_{\dot{\alpha}}\bar{D}'^{\dot{\alpha}}\delta(z-z')=-D^2\bar{D}'^{\dot{\alpha}}\bar{D}'_{\dot{\alpha}}\delta(z-z')\\&=:D^2\bar{D}'^2:\ \delta(z-z')=\bar{D}'^2D^2\delta(z-z')=\bar{D}'^2D'^2\delta(z-z')\end{aligned}$$

where in the last step we used (28c). With reasoning similar to this, we see that when acting on $\delta(z-z')$ the order of Grassmann differential operators is reversed when the differentiation is switched from z to z'. However, the order at which these differential operators act on some function $F(z')$ is restored after the integration by parts is carried out, which we illustrate in the simple example

$$\int d^8z'\, G(z)\bar{D}^2D^2\delta(z-z')F(z')$$

$$=\int d^8z'\, G(z)[D'^2\bar{D}'^2\delta(z-z')]F(z')$$

$$=\int d^8z'\, G(z)\delta(z-z')\bar{D}'^2D'^2\delta(z-z')$$

4. DISCUSSION AND CONCLUSION

As we see, the normal-ordered product of θ's and $\bar{\theta}$'s is simply their ordinary product. The normal-ordered product of θ's, $\bar{\theta}$'s, and the anticommuting Grassmann differentical operators can again be expressed as an ordinary product in which all the differential operators stand to the right of θ's and $\bar{\theta}$'s. However, the normal-ordered product involving θ's, $\bar{\theta}$'s, and nonanticommuting Grassmann differential operators cannot generally be expressed in terms of ordinary products. This peculiarity, of course, arises because nonanticommuting Grassmann differential operators have contractions between them. From the practial point of view, however, this is no handicap, since in practical applications all these normal products act on some quantity that is only a function of x; now, under a normal-ordered product differential operators D_α and $\bar{D}_{\dot{\alpha}}$ can be replaced with $-_\alpha(\sigma^m\bar{\theta})P_m$ and $(\theta\sigma^m)_{\dot{\alpha}}P_m$, respectively.

In conclusion, we can say that the concept of the normal-ordered Grassmann operators not only has some new elements not found in normal-ordered quantum fields in field theory, but also can be very useful in practical evaluations in supersymmetry.

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